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POINT, LINE, AND SURFACE, AS SETS OF SOLIDS [1]

THE following pages contain a series of definitions of geometrical concepts, based upon the assumed entity "solid" and the assumed relation "can connect." The precise meaning to be attached to these fundamental terms could only be made clear by a set of geometrical postulates in which they were involved; but no such set will be provided. In explanation I would say that the object in view is not the construction of a geometry, but the bringing to light of a certain limited order of relationships. It is the possibility of a geometry, rather than the geometry itself that is to be exhibited.

The formal explanation of the assumed terms being omitted, it is necessary to give an informal explanation of them. The "solid," then, may be said to be *the space occupied by a physical solid.* This, of course, is no definition—the definition of a term assumed as fundamental is in any case out of the question—but a suggestion to the reader as to what the writer is thinking about. In the same way it may be said that the assumed solids are of all possible shapes, but that no acquaintance with any particular shape is to be taken for granted. Professor Huntington's remarkable system of geometry, in which the *sphere* is assumed as an indefinable, will sufficiently illustrate what the system here suggested is not.

The statement, that the "solids" may be of all possible shapes, is to be understood as subject to this limitation: the shape must in each case be one which a single physical solid can be conceived to have. The limitation is not absolutely necessary for our purposes. With a little modification the series of definitions would stand, even if we admitted the possibility that a particular "solid" might consist of a number of wholly disconnected parts; and without change in the definitions we might admit solids consisting of parts that are in contact only at points or along lines,—like a double cone whose parts are connected only at the vertex, or a combination of two cubes that have a common edge. But no real increase in generality would be gained in this way; and the interests of simplicity would seem to recommend the limitation.

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[1] This article may be regarded as an appendix to the account of the *Nature of Space,* published in recent numbers of this JOURNAL.

We shall have no occasion to make any distinction between a possible and an actual solid; every possible solid will be treated as actual. Consequently we shall meet with all manner of intersection of solids. They will pass into and through one another, as physical solids do not. It is true that parts of one and the same physical solid may be conceived as being each a physical solid; and these parts may interpenetrate to our hearts' content. But in the case of the solids with which we shall have to deal, we shall suppose that no matter which two of them be considered, there is a third that intersects them both, or even includes them both as parts.

With respect to their magnitude, it is to be said that all the solids that we shall assume are finite, and that none are of zero magnitude. We shall have no occasion either to affirm or to deny the existence of infinite solids. We shall assume that there are no zero solids. The contrary assumption might be made. But it would call for some changes in the definitions; though these would amount to a less profound modification of the system than might easily be supposed.

The solids which we are to consider are not endowed with motion. They are to be conceived as eternal beings. Our "constructions" will be literally no constructions: we shall but turn our attention to what was already there. Nevertheless, the relations that we shall study, and in particular the fundamental relation "can connect," could scarcely have been suggested but by the behavior of movable physical solids. More explicitly, if the physical solids *A* and *B* are at rest (relatively to the field of observation), and the solid *C* is such that we can manipulate it with a fair degree of freedom, to say that *C* can connect *A* and *B* would be understood to mean that we could, if we wished, put *C* in simultaneous contact with *A* and *B*; and it is, so to speak, the shadow of this physical relation that we shall assume for our geometrical solids.

Yet there are some noteworthy modifications of the relation, due especially to the free interpenetration of our solids. This will be made clear if, provisionally, we define the geometrical "can connect" in other terms. "*A* can connect *B* and *C*" is to be understood as meaning that there exists at least one solid *X*, such that *X* is equal in all its parts to *A*, and such that *X* has at least one point in common with *B* and at least one point in common with *C*. In other words, the "connecting" may be done *either by overlapping or by external contact.*

Accordingly, the physical relation "can connect" may easily fail to obtain, where in the analogous case the geometrical relation would obtain. Suppose the physical solid *B* to be hollow; and suppose that *C* lies deep inside *B*. Then no physical solid *A*, that lies beside *B*, can possibly be made to connect *B* and *C*. But in the analogous case for geometrical solids, if *A* is only big enough, there is no trouble about its being "able" to do the connecting. Suppose further that the hollow physical solid *B* not only has *C* inside it but wraps it tightly around, so that they are in contact over the whole surface of *C*. Then there will, in general, be *no* physical solid that can be brought into simultaneous contact with them. In the analogous case for geometrical solids, *any* solid "could" do the connecting.

In the course of the development, a modified form of Professor Whitehead's method of "extensive abstraction" is introduced. The modification consists in the use, not of the relation of "extending-over" (the relation of whole to part), but of the relation of "containing," in the sense of not simply including as a part but completely enveloping. Through this modification the method is greatly simplified and strengthened. It is, I believe, impossible by means of the method in its original form to give a definition of the point in terms of the solid.[2]

If containing is assumed as a primitive relation, extending-over can easily be defined. Thus, to speak of solids, "*A* extends over *B*" can be explained as meaning: "There is a solid which is contained by *A* but not by *B*, and there is no solid that is contained by *B* but not by *A*." Hence the defining-power of the modified method is not inferior to that of the original. It is, in fact, much greater.

However, in the present treatment neither containing nor extending-over is assumed as primitive, but both are defined in terms of the relation "can connect." This mode of approach has the further advantage, that the conceptions of *length* and *collinearity* can be defined without the introduction of any additional indefinable. A complete conceptual foundation for geometry is thus provided.

With the definitions two postulates are included, which are of special importance for understanding the real significance of the definitions.

INDEFINABLES

Solid.—(Solids are to be denoted by capital letters. Different letters need not denote distinct solids. Classes of solids will, till further notice, be denoted by small letters.)

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[2] *Cf.* A. N. Whitehead, *The Concept of Nature*, p. 86f.; also a note by the present writer in the *Philosophical Review* for March, 1921. The definition of the point which I once offered as an illustration of Professor Whitehead's method (in a review of his "Principles of Natural Knowledge" in the *Philosophical Review* for May, 1920) involves both an error of interpretation and a serious blunder.

Can connect.

Postulates

Postulate of Identity: If *A* and *B* are such that there are no *W* and *X*, such that *W* can connect *A* and *X* but can not connect *B* and *X*; and, similarly, there are no *Y* and *Z*, such that *Y* can connect *B* and *Z* but can not connect *A* and *Z*; *A* and *B* are identical.

In other words, if *A* and *B* are alike in their capacity of being connected with other solids, they are identical.

Postulate of Measurement: If *A* and *B* are such that *W* and *X* exist, such that *A* can connect *W* and *X* but *B* can not, then there are no *Y* and *Z* such that *B* can connect *Y* and *Z*, but *A* can not.

When we have defined the expression "longer than" we may restate this principle in the form: If *A* is longer than *B*, *B* is not longer than *A*.

DEFINITIONS

I. If *A* and *B* are such that *X* exists, such that *X* can not connect *A* and *B*, *A* and *B* are said to be *disconnected.*

II. If *A* and *B* are not disconnected, they are said to be *connected.*

III. If *A* and *B* are such that every solid connected with *A* is connected with *B*, but not every solid connected with *B* is connected with *A*, *A* is said to be a *part* of *B*.[3]

IV. If *A* and *B* are not identical and have a common part, they are said to *intersect.*[4]

V. If *A* and *B* have no common part they are said to be *separated.*

The reader should note carefully the distinction between the terms "disconnected" and "separated." Any two solids that are disconnected are separated; but the converse is not true.

VI. If *A* and *B* are separated, but not disconnected, they are said to be *in contact.*

VII. If *A* is a part of *B*, and if every solid that is in contact with *A* intersects *B*, *A* is said to be *contained* in *B*.

We are now ready to proceed to the definition of the point by the method of extensive abstraction.

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[3] It is worthy of remark that the relation of whole and part can also be defined directly in terms of the relation can-connect. Thus we may say: If *A* and *B* are not identical, and if there are no *X* and *Y* such that *X* can connect *A* and *Y* and can not connect *B* and *Y*, *A* is said to be a part of *B*.

[4] Here, and as often as possible below, I follow Professor Whitehead's terminology.

VIII. If a set of solids is such that

(1) of every two members of the set, one is contained in the other;

(2) there is no solid contained by every member of the set;

it is called an *abstractive set.*[5]

The first specification shows that the abstractive set is like a *nest of boxes,* one within another; except, to be sure, that whereas each smaller box lies within the hollow interior of the larger box, each smaller solid of the abstractive set is a part of the larger solid. The second specification shows that the abstractive set is an unending sequence of solids. There can not be a smallest solid of the set; because, if there were, any solid which it contained would be contained by all the solids of the set.

In ordinary terms we would say that the abstractive set must converge upon a point, a line or curve, a surface, or some combination of lines and surfaces. We have not assumed the existence of such entities as points, lines, and surfaces, and so can not use them for the definition or classification of abstractive sets. It will be shown how abstractive sets may be used for the definition of points, lines, and surfaces.

As an example of an abstractive set, the reader may consider a set of concentric spheres, growing less and less *ad infinitum*—never reaching nothingness but approaching it as a limit. Or, in place of the spheres, we might have, say, concentric cubes. Again, we might have a set of co-axial cylinders, diminishing in such a way that the radius approaches zero, while the altitude approaches a limit which is not zero. Yet again, we might have a set of rectangular parallelopipeds, diminishing so that one dimension approaches zero, while each of the other dimensions approaches a limit which is not zero.

IX. If *m* is an abstractive set, the class of the solids that contain members of *m* is called an *abstractive element,* or simply an *element.*

X. If two abstractive elements are not identical, and one logically includes the other, the former is said to *lie in* the latter.

Note that different abstractive sets may serve to define the same abstractive element. To return to the above examples, if the set of the concentric cubes and the set of concentric spheres have the same center, every solid that contains one of the cubes will contain one of the spheres; and conversely, every solid that contains a sphere will contain a cube. Thus the set of cubes and the set of spheres determine the same element.

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[5] This definition departs from Professor Whitehead's by substituting the relation of containing for that of including as a part. There is the further difference, that the sets with which he deals are not sets of solids but sets of four-dimensional *events.*

On the other hand, consider the case of a set of co-axial cylinders, such as was suggested above, and a set of concentric spheres whose center lies in the midst of the common axis of the cylinders. Every solid that contains one of the cylinders will contain one of the spheres; but there will be solids that contain some of the smaller spheres but do not contain any of the cylinders. Hence the class of the solids that contain members of the set of spheres logically includes the class of the solids that contain members of the set of cylinders. It is in such a case that, in accordance with Definition *X*, we say that the one class lies in the other.

XI. A *point* is an abstractive element in which no other abstractive element lies.

It will be observed that the point is here defined as the class of those solids which would ordinarily be described as containing the point—that is to say, the point would lie in each solid, but not in its surface. The formal properties of the point as here defined are easily seen to be identical with those of the point conceived as an irreducible individual.

The following definitions are much less important and increase rapidly in difficulty. The reader who so desires may pass at once to the concluding remarks.

Hereafter abstractive elements, as well as solids, will be denoted by capital letters.

XII. If the solid A is a member of the point P, P is said to be *contained* in A.

This use of "contained" will be found to be closely analogous to its use as denoting a relation between two solids. Thus no serious ambiguity is involved.

XIII. If every solid that contains the solid A contains the point P, P is said to *lie* in A.

This definition may be extended so as to embrace the analogous relation between a solid and any abstractive element. The present definition may also be stated: If the class of solids that contain the solid A is logically included in the point P, P is said to lie in A.[6]

It will be observed that the relation between solid and point (or other element), which is here defined, is closely analogous to the relation "to lie in" subsisting between two abstractive elements (Definition *X*).

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[6] An alternative definition that is worthy of notice is this: If no solid that contains the point P is disconnected with the solid A, P is said to lie in A.

XIV. If the point P is a member of the set of points m, and there is a solid that contains P and contains no other member of m, P is said to be *isolated* in m.

Note that if m has but one member, that is an isolated member.

XV. If a point P is not a member of the set of points m, but every solid that contains P contains members of m, P is said to be *adjacent* to m.

Consider, for example, the set of the points that lie in a straight line PQ, between the extremities P and Q. Both P and Q are adjacent to this set.

XVI. If a set of points m is included in a set n, and there is a member of m which is not adjacent to the set of those points which are members of n but not of m, m is said to be *divergent* in n.

Thus suppose n is the set which includes the points within a circle, those in its circumference, and those in a line tangent to the circle; and suppose m is the set of the points in the tangent. Then, except the point of tangency, no member of m is adjacent to the set of the points that are in n but not in m; and accordingly m is divergent in n.

XVII. If a set of points includes no isolated points, and has no points adjacent to it, it is said to be *perfect.*

This definition is substantially in accord with Cantor's metrical definition of the term. What it really amounts to depends, of course, on the existential postulates that determine the universe of solids. It should be observed that a perfect set may consist of several parts that are in no wise connected with one another.

XVIII. If a perfect set is such that every perfect set which it includes is divergent in it, it is said to be a *one-dimensional* set.

XIX. If a perfect set is such that no one-dimensional set which it includes is divergent in it, it is said to be a *doubly perfect* set.

XX. If a doubly perfect set is such that every doubly perfect set which it includes is divergent in it, it is said to be a *two-dimensional* set.

XXI. If a doubly perfect set is such that no two-dimensional set which it includes is divergent in it, it is said to be a *trebly perfect* set.

XXII. If a trebly perfect set is such that every trebly perfect set which it includes is divergent in it, it is said to be a *three-dimensional* set.[7]

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[7] While the order of Definitions XVII–XXII is fixed, the statement of them may be conveniently consolidated as follows:

$$\text{If a set of points}\left\{\begin{array}{c}\text{has no point adjacent to it}\\ \text{is perfect}\\ \text{is doubly perfect}\end{array}\right.\quad\text{and includes no}\left\{\begin{array}{r}\text{isolated}\\ \text{divergent}\\ \text{divergent}\end{array}\right.$$

It should be observed that if the set of all points is three-dimensional, every trebly perfect set is three-dimensional.

XXIII. If a $\left\{\begin{matrix}\text{perfect}\\ \text{doubly perfect}\\ \text{trebly perfect}\end{matrix}\right.$ set of points is such that it includes no two $\left\{\begin{matrix}\text{perfect}\\ \text{doubly perfect}\\ \text{trebly perfect}\end{matrix}\right.$ sets that exhaust its members and themselves include no common $\left\{\begin{matrix}\text{point,}\\ \text{perfect set,}\\ \text{doubly perfect set,}\end{matrix}\right.$ it is said to be $\left\{\begin{matrix}\textit{continuous.}\\ \textit{doubly continuous.}\\ \textit{trebly continuous.}\end{matrix}\right.$

XXIV. If a $\left\{\begin{matrix}\text{one-dimensional}\\ \text{two-dimensional}\\ \text{three-dimensional}\end{matrix}\right.$ set of points is $\left\{\begin{matrix}\text{continuous,}\\ \text{doubly continuous,}\\ \text{trebly continuous,}\end{matrix}\right.$ it is said to be a $\left\{\begin{matrix}\textit{one-dimensionally continuous}\\ \textit{two-dimensionally continuous}\\ \textit{three-dimensionally continuous}\end{matrix}\right.$ set.

The two following definitions are somewhat aside from the purpose of the present discussion; but they are given here because of their utility in serving to define various types of lines and surfaces.

XXV. If two sets of points have no common member, and are such that there is no continuous set that includes members of both but no other points, the two sets are said to be *disjoined.*

XXVI. If a set of points *p* (which may consist of one point *P*) and a continuous set *m* have a common member or members, and the remaining members of *m* consist of two disjoined sets, *p* is said to *divide m*; and if, further, no set included in *p* (not identical with it) divides *m, p* is said to divide *m economically.*

One-dimensionally continuous sets may now be classified according as they are divided (i) by any one of their points; (ii) by any one of their points except one; (iii) by any except two; (iv) by any two, but by no one; *etc.* Similarly, various types of two-dimensional sets may be characterized by the number and type of the one-dimensional sets that divide them.

It has become common in recent years to regard the line and the surface—and, indeed, the solid also—as sets of points. From

$\left\{\begin{matrix}\text{point,}\\ \text{one-dimensional set,}\\ \text{two-dimensional set,}\end{matrix}\right.$ it it said to be $\left\{\begin{matrix}\textit{perfect.}\\ \textit{doubly perfect.}\\ \textit{trebly perfect.}\end{matrix}\right.$

If a set of points is $\left\{\begin{matrix}\text{perfect,}\\ \text{doubly perfect,}\\ \text{trebly perfect,}\end{matrix}\right.$ and every $\left\{\begin{matrix}\text{perfect}\\ \text{doubly perfect}\\ \text{trebly perfect}\end{matrix}\right.$ set which it includes is divergent in it, it is said to be a $\left\{\begin{matrix}\textit{one-dimensional}\\ \textit{two-dimensional}\\ \textit{three-dimensional}\end{matrix}\right.$ set.

that standpoint, the definition of the one-dimensionally continuous set may be taken as the definition of the line; and the definition of the two-dimensionally continuous set (or, if it be preferred, the continuous two-dimensional set) may be taken as that of the surface. Professor Whitehead, however, has shown that the line and the surface may be regarded as abstractive elements. The following definitions apply to them in that capacity.

XXVII. If an abstractive element is such that the set of the points which lie in it is one-dimensional, it is called a *line.*

XXVIII. If an abstractive element is such that the set of the points which lie in it is two-dimensional, it is called a *surface*; and if, further, the set is doubly continuous, the surface is said to be *unitary.*[8]

Does the figure *8* bound one surface or two? According to this definition, it bounds one surface, but that is not a unitary surface.

We now proceed to some metrical conceptions. The development follows closely the lines laid down in the second article of this series (pp. 428–9). In some cases no change in the definitions there given is called for; in other cases, only the omission of all reference to "chains" of solids. The reader is, therefore, hereby referred to the earlier discussion. In the case of the definition of the sum of two lengths, however, a more serious revision is necessary.

XXIX (*a*) *Farther apart.*	(*b*) *Longer.*
XXX (*a*) *Just as far apart.*	(*b*) *Just as long.*
XXXI (*a*) *Distance.*	(*b*) *Length.*
XXXII (*a*) *Greater* (of distances).	(*b*) *Greater* (of lengths).
XXXIII (*a*) *Equal* (distance to length).	(*b*) *Equal* (length to distance).
XXXIV (*a*) *Greater* (distance compared to length).	(*b*) *Greater* (length compared to distance).

XXXV. If A and B are such that there is no solid X such that X can not connect A and B, the distance between A and B is said to be *zero.*

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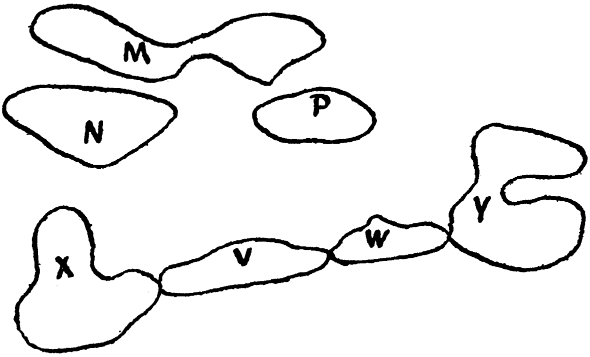
[8] These definitions are somewhat wider than those which Professor Whitehead has given, and I believe are better in accord with the tradition and the needs of the science. For example, according to Professor Whitehead, there is no complete spherical surface except in the sense of a set of points. The reason is that Professor Whitehead is hampered by limitations upon the possible shape of the four-dimensional entities—events—which his system uses as its basis. (*Cf. The Concept of Nature,* pp. 101ff.)

In many cases *hollow* solids are required for the abstractive sets upon which surfaces are founded. That is the case with the spherical surface, for example. Similarly, for the abstractive sets upon which lines are based, *ring-shaped* solids are often needed. But such cases offer no peculiar difficulty.

This is the relation that was earlier expressed by saying that *A* and *B* were *connected.* We might define a zero-length as the length of a solid which can not connect any solids except those the distance between which is zero. But we are not to assume the existence of such solids. If we did, we should very naturally reserve for them the name of "points."



XXXVI. (*b*) If *M, N,* and *P* are such that *X* and *Y* exist, such that the distance between *X* and *Y* is equal to the length of *M,* and such that *V* and *W* exist such that *V* is just as long as *N,* and *W* is just as long as *P*; and if the distances between *X* and *V, V* and *W,* and *W* and *Y* are zero; and if no solid *M′* exists, such that *M′* is longer than *M* and stands in this same relation to *N* and *P*; the length of *M* is said to be the *sum* of the lengths of *N* and *P.*

XXXVI. (*a*) *Sum* (of two distances).

In the same way, we may define the sum of any number of lengths or of distances, or of both lengths and distances. The sum of a length and a distance may, by convention, be regarded as a distance.[9]

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[9] Attention may be called to the fact that these metrical definitions are entirely independent of those which precede them in the present series, for they go back directly to the indefinable relation, "can connect." Accordingly, it is possible, and for some purposes it may be advantageous, to substitute metrical definitions of the relations of whole and part, containing, *etc.,* and these deserve a passing mention.

(In place of Definitions II and I.) If the distance between *A* and *B* is zero, they are said to be *connected;* otherwise they are said to be *disconnected.*

(In place of Definition III.) If *A*ˡ and *B* are such that *X* does not exist such that the distance between *B* and *X* is greater than the distance between *A* and *X;* and if *Y* does exist such that the distance between *A* and *Y* is greater than the distance between *B* and *Y*; *A* is said to be a *part* of *B.*

In other words, if no solid is farther from *B* than from *A,* and there is a solid that is farther from *A* than from *B, A* is a part of *B.*

The distance between a point and a solid, and the distance between two points require special treatment.

XXXVII. If the point A and the solids B and C are such that C can connect B and any member of A, but for all values of C', where C' is a solid and C is longer than C', there is a member of A which C' can not connect with B; the *distance* between A and B is said to be the distance which is equal to the length of C.

XXXVIII. If the points A and B and the solid C are such that C can connect any member of A and any member of B; but for all values of C', where C' is a solid and C is longer than C', there is a member of A and a member of B which C' can not connect; the *distance* between A and B is said to be the distance that is equal to the length of C.

The definition of the between-relation and of collinearity is now effected as in the first article of this series.

XXXIX. If the points A, B, and C are distinct, and are such that the distance between A and C is the sum of the distances between A and B and between B and C; B is said to be *between* A and C.

XL. If either the point A is between the points B and C, or B is between A and C, or C is between A and B, the three points are said to be *collinear.*

The straight line, as an abstractive element, may now be defined.

XLI. A *straight* line is a line such that every three points that lie in it are collinear.

Are there straight lines of infinite length? That depends upon the question whether there are solids of infinite length; which in a Euclidean geometry is as much as to ask whether there are solids that can connect any two solids whatsoever. It is of very slight importance whether an affirmative or a negative answer is assumed.

A matter of far greater importance is the "fixing" of the straight line by any two of its points. This amounts to the proposition, that if the points A, B, and C are collinear, and the points A, B, and D are collinear, then A, C, and D are collinear. In the choice of postulates upon which a system of geometry is to be founded, this is one of the essential aims to be held in view.

(In place of Definition VII.) If A and B are such that no X exists such that X is separated from A and such that the distance between A and X is not greater than the distance between B and X; A is said to *contain* B.

The definition of abstractive sets and elements may be left unchanged. The following suggests itself as the appropriate metrical definition of the point:

(In place of Definition XI.) If an abstractive element A is such that for every solid X there is a member of A that is not longer than X, A is called a *point.*

What, now, is the philosophical significance of this, or an equivalent, series of definitions. And, in particular, what significance has the identification of the point with a class of solids? I believe it to be considerable, but that it is open to serious misinterpretation.

Recent theories with respect to the relation between geometry and mechanics have given rise to a demand for the definition of the point in definitely experiential terms. For points are not primary data of experience—if there are such data. They are never perceived by us, and their existence is never made evident by any distinguishable effect, however delicate, of their presence. We can not infer their existence from the perturbation of the orbit either of a planet or of an electron. They are conceptual constructs; and it is a problem for analytical science to exhibit the mode of their construction.

On its face, the method of extensive abstraction is an application of "Occam's razor." Instead of the point, which we do not perceive, we are given a class of solids such as we do perceive; and thus entities are not multiplied unnecessarily. But the point, as we find it in geometrical tradition, is not an altogether distinct kind of entity. It is a solid, remarkable in only one fundamental respect, namely, that its length is zero. To be sure, no solids of zero length are perceived by us; but we could not perceive them if they existed. And, furthermore, although we perceive solids, we perceive no abstractive sets of solids; and there is no indirect empirical assurance that such sets exist—only suggestive evidence that entitles us to assume that they exist. In accepting the abstractive set, we are as veritably going beyond experience as in accepting the solid of zero-length.

It may be replied that the assumption of the abstractive set is in any case more economical than that of the zero-solid, because if there are zero-solids there are abstractive sets, while there may be abstractive sets without zero-solids. But this statement, I believe, is only superficially correct. Just because the zero-solid is an entity that lies beyond the limits of any possible direct or indirect perception, the assumption of its existence means less than the formal proposition indicates. A real point (as I have elsewhere had occasion to urge) means no more or less to us than a possible point. Hence the method of extensive abstraction has not so much eliminated the zero-solid as it has analyzed it. The method has made us realize more distinctly than ever before what the assumption of the zero-solid logically amounts to.

Thus, if I am right, the method of extensive abstraction simply gives us one more illustration of Galileo's great principle: that the laws of physics—among which the laws of geometry may here be included—are descriptions of ideally simple cases, which no experience presents to us, but which the objects of our experience do with various degrees of closeness approach. The formal elimination of the limit itself, in the case of geometry, and the statement of the laws in terms of an infinitely continued approximation, only brings out with a new clearness what their real nature has always been. I say it does only this; but is not that sufficient?

Meanwhile we ought not to forget that the geometrical solid itself is not given in experience.[10] It, too, is the product of an idealization —if not individually, then as a member of its class. If we say that the physical solid is a geometrical solid and more, we forget that no perception assures us that it has the most elementary properties of a geometrical solid; for those properties are relative to the existence of other geometrical solids which are *not* physical solids. As of the point, so we must say of the geometrical solid itself: the distinction between the possible and the actual has no place. To define the point as a class of solids is *not* to find a place for it in the real world. That can only be done by analyzing those properties of the physical solid upon which geometry, as an empirical science, is founded.

If the point can be conceived as a set of solids, so the solid—the geometrical solid—can be conceived as a set of points. So far as the formal logical relations are concerned, there is not a particle of advantage in the matter, either on the one side or on the other. Historically, as I have elsewhere urged, the conception of space had to wait upon the development of the point. Psychologically, the point has this advantage over the geometrical solid: that its very smallness accounts sufficiently for its absolute imperceptibility, and it is thus able to serve as a middle term for the thought-transition from the physical to the geometrical solid. Practically, it is the point that gives space its excuse for being. A space without points would be little more than an obstacle between us and the physical world.

Every new scientific perspective is valuable; and the method of extensive abstraction has given us a new perspective of very great value indeed. But we must not let ourselves fall into the illusion that the novel order which it presents is truer, or necessarily more fundamental, than that which has long been familiar to us.

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[10] I should, of course, say the same of Professor Whitehead's "events."